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THE AVERAGE FIRST-RECURRENCE TIME

by

FC

Bernard Friedman and Ivan Niven

Technical Report No. 3

Prepared under Contract Nonr-222(37)

(NR 041 157)

For

Office of Naval Research

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**Department of Mathematics
University of California
Berkeley 4, California**

July, 1956

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Contents

	<u>Page</u>
Introduction.	1
The One-Dimensional Case.	3
Bibliography.	18

The Average First Recurrence Time

Introduction

The kinetic theory of gases contains a fundamental paradox in that the results obtained from the theory are irreversible despite the fact that gases are assumed to be composed of molecules which obey the reversible laws of mechanics. The paradox is further heightened by Poincaré's Cycle Theorem which states roughly that, no matter what is the initial configuration of the molecules, the gas must return to a state which is arbitrarily close to the initial state and that it must do so infinitely often. Physicists have tried to explain this paradox by stating that the time for one such Poincaré cycle is so large that the recurrence of an initially improbable state is unlikely during the times normally available for observation. For a fuller discussion of these topics we refer to an article by Chandrasekhar¹.

In the present paper we consider the first recurrence time of a very simple dynamical system, namely, one which has $k+1$ degrees of freedom x_1, x_2, \dots, x_{k+1} , each of which is simply-periodic in the time τ . Put

$$x_j(\tau) = A_j e^{2\pi i v_j \tau}, \quad j=1, 2, \dots, k+1;$$

then the problem is to determine the smallest positive value of τ such that the point $(x_1(\tau), \dots, x_{k+1}(\tau))$ will be within a preassigned neighborhood of the initial point $(x_1(0), \dots, x_{k+1}(0))$.

For simplicity, the neighborhood is assumed to be such that

$$(1) \quad |v_j\tau - m_j| < \epsilon, \quad j=1, 2, \dots, k+1,$$

where m_1, \dots, m_{k+1} are integers. The smallest positive value of τ such that the inequalities (1) are satisfied for some set of integers will be called the first recurrence time of the system.

Note that

$$\tau = \tau(v_1, \dots, v_{k+1}; \epsilon).$$

We wish to find the average of the first recurrence time for all such systems, that is,

$$(2) \quad \tau_{av} = \int_0^1 \dots \int_0^1 \tau(v_1, \dots, v_{k+1}; \epsilon) dv_1 \dots dv_{k+1}$$

and in particular the asymptotic behavior of τ_{av} as ϵ approaches zero.

Suppose that v_{k+1} is the largest of the v_j and instead of the inequalities (1) consider the following:

$$(3) \quad v_{k+1}\tau - m_{k+1} = 0, \quad |v_j\tau - m_j| < \epsilon, \quad j=1, 2, \dots, k.$$

Put $a_j = v_j/v_{k+1}$, $m_{k+1} = t$, then (3) becomes

$$(4) \quad |ta_j - m_j| < \epsilon, \quad j=1, 2, \dots, k,$$

where t, m_1, \dots, m_k are integers. We put

$$t_{av} = \int_0^1 \dots \int_0^1 t(a_1, \dots, a_k; \epsilon) da_1 \dots da_k.$$

We shall show that there exist two constants c_1 and c_2 such that

$$(5) \quad c_1 \epsilon^{-k} < t_{av} < c_2 \epsilon^{-k}$$

For the case $k=1$, more can be obtained. We prove then that

$$(6) \quad t_{av} = \frac{6}{\pi^2} \log 2 \epsilon^{-1} + O(\epsilon^{-1/2}).$$

The One-Dimensional Case

For any α in the unit interval, define $t = t(\alpha, \epsilon)$ as the smallest positive integer such that $|t\alpha - m| \leq \epsilon$ for some integer m , i.e., such that $t\alpha$ is within ϵ of an integer. Our problem is to evaluate

$$(7) \quad \int_0^1 t(\alpha, \epsilon) d\alpha.$$

Let the integer n be chosen so that $\frac{1}{n} \leq \epsilon < \frac{1}{n+1}$. Then it is

clear that for any α ,

$$t(\alpha, \frac{1}{n}) \leq t(\alpha, \epsilon) \leq t(\alpha, \frac{1}{n+1})$$

and so

$$\int_0^1 t(\alpha, \frac{1}{n}) d\alpha \leq \int_0^1 t(\alpha, \epsilon) d\alpha \leq \int_0^1 t(\alpha, \frac{1}{n+1}) d\alpha.$$

Thus we examine

$$\int_0^1 t(\alpha, \frac{1}{n}) d\alpha.$$

Any real number α is said to have an admissible value k (a positive integer) if $k\alpha$ is within $1/n$ of an integer. Thus $t(\alpha, 1/n)$ is the minimum of the admissible values for α . Any rational number a/b with $0 \leq a/b \leq 1$ defines an interval

$$I(a/b) = \left(\frac{a}{b} - \frac{1}{nb}, \frac{a}{b} + \frac{1}{nb} \right),$$

and all points in $I(a/b)$ have admissible value b . Furthermore b is an admissible value for no real values of α other than those in the intervals $I(0/b), I(1/b), I(2/b), \dots, I(b/b)$.

Now if t is the minimum integer such that $t\alpha$ is within $1/n$ of an integer m , then $t\alpha - m = \frac{\delta}{n}$ with $|\delta| \leq 1$. Thus $\alpha - m/t = \frac{\delta}{tn}$, and m/t is in its lowest terms since otherwise $t/(t,m)$ would be an admissible value for α smaller than t itself. Thus $\alpha \in I(m/t)$.

Let F_n denote the Farey series of order n ,

$$F_n: \frac{0}{1}, \frac{1}{n}, \dots, \frac{n-1}{n}, \frac{1}{1},$$

the series of rational numbers a/b in ascending order satisfying $0 \leq a \leq b \leq n$ with $(a,b) = 1$.

Now if the Farey series F_{n-1} , of fractions a/b say, is used to define a collection C of intervals $I(a/b)$, we shall prove that C covers the unit interval, but that no single interval $I(a/b)$ contains any member of F_{n-1} other than a/b . From this it follows that to find $t(\alpha, 1/n)$ we need merely locate a

between two consecutive fractions $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ of F_{n-1} and we have

$$t(a, 1/n) = b_1 \text{ if } a \text{ is in } I\left(\frac{a_1}{b_1}\right) \text{ but not in } I\left(\frac{a_2}{b_2}\right);$$

$$t(a, 1/n) = b_2 \text{ if } a \text{ is in } I\left(\frac{a_2}{b_2}\right) \text{ but not in } I\left(\frac{a_1}{b_1}\right);$$

$$t(a, 1/n) = \min(b_1, b_2) \text{ if } a \text{ is in both } I\left(\frac{a_1}{b_1}\right) \text{ and } I\left(\frac{a_2}{b_2}\right).$$

To prove the assertions above about the collection of intervals C : If $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are adjacent fractions in F_{n-1}

then $\frac{a_1}{b_1} - \frac{a_2}{b_2} = \pm \frac{1}{b_1 b_2}$ and since $b_2 < n$ then $\frac{a_2}{b_2}$ is not in $I\left(\frac{a_1}{b_1}\right)$. To prove that $I\left(\frac{a_1}{b_1}\right)$ and $I\left(\frac{a_2}{b_2}\right)$ meet, (which will

establish that C covers the unit interval) we must prove that

$$\frac{1}{nb_1} + \frac{1}{nb_2} \geq \frac{1}{b_1 b_2}. \text{ This follows from the fact that}$$

$$b_1 + b_2 \geq n.$$

From this argument we see that the interval $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$, of length $\frac{1}{b_1 b_2}$, splits into two parts as regards the value of

$$t(a, \frac{1}{n}):$$

$$t(a, \frac{1}{n}) = \min(b_1, b_2) \text{ over a subinterval of length } \frac{1}{n \cdot \min(b_1, b_2)}$$

$$t(a, \frac{1}{n}) = \max(b_1, b_2) \text{ over one of length } \frac{1}{b_1 b_2} - \frac{1}{n \cdot \min(b_1, b_2)}.$$

Thus

$$\begin{aligned} \int_{a_1/b_1}^{a_2/b_2} t(a, \frac{1}{n}) da &= \frac{\min(b_1, b_2)}{n \cdot \min(b_1, b_2)} + \frac{\max(b_1, b_2)}{b_1 b_2} - \frac{\max(b_1, b_2)}{n \cdot \min(b_1, b_2)} \\ &= \frac{1}{n} + \frac{n - \max(b_1, b_2)}{n \cdot \min(b_1, b_2)} \end{aligned}$$

Let us denote $\min(b_1, b_2)$ by b and $\max(b_1, b_2)$ by B , so that

$$(8) \quad \int_0^1 t(a, \frac{1}{n}) da = \sum \left(\frac{1}{n} + \frac{n-B}{nb} \right)$$

where the sum ranges over all pairs of consecutive fractions in F_{n-1} .

Lemma 1. Consider the set S_b of all fractions in F_{n-1} with fixed denominator b , where $1 \leq b \leq n-2$. Let T be the set of all fractions which are neighbors to members of S_b , and which have denominators exceeding b . If B is any integer such that $(B, b) = 1$, $b + B \geq n$, $n-1 \geq B \geq b + 1$, then the set T has exactly two members with denominator B .

Proof. The equation $xb - yB = 1$ has exactly one solution in integers x and y such that $1 \leq x \leq B-1$, $1 \leq y \leq b-1$. The fractions x/B and y/b are in F_{n-1} , and are neighbors, for if y/b had the neighbor h/k on the right, $y/b < h/k < x/B$ then $hb - yk = 1$ and $xb - yB = 1$ imply that

$$b(xk - hB) = k - B.$$

But $xk - hB \geq 1$, so that $k - B \geq b$ and $k \geq b + B \geq n$, a contradiction.

Similarly the equation $Xb - YB = -1$ has exactly one solution X, Y with $1 \leq X \leq B-1$, $1 \leq Y \leq b-1$ and likewise the fractions $X/B < Y/b$ are neighbors in F_{n-1} . Thus we have shown that the set T contains at least two fractions with denominator B . On the other hand if a/b and A/B are neighbors in F_{n-1} , then $aB - Ab = \pm 1$, so that the two cases we have examined are the only possible ones. Hence the lemma is proved.

Returning to the formula (8), we first compute the sum $\sum 1/n$. Let $\varphi(k)$ be the Euler function denoting the number of integers less than and prime to k , and define

$$\bar{\Psi}(n) = \varphi(1) + \varphi(2) + \dots + \varphi(n).$$

Then the number of terms in F_{n-1} with denominator $k \leq n-1$ is $\varphi(k)$, except in case $k = 1$, there being two terms with denominator 1. Hence we have

$$\sum \frac{1}{n} = \frac{1}{n} \left\{ \varphi(1) + \varphi(2) + \dots + \varphi(n-1) \right\} = \frac{1}{n} \bar{\Psi}(n-1).$$

By virtue of this and Lemma 1 we see that (8) can be written as

$$(9) \quad \int_0^1 t(a, \frac{1}{n}) da = \frac{1}{n} \bar{\Psi}(n-1) + \frac{2}{n} \sum_{b=1}^{n-2} \sum_{\substack{B=b+1 \\ B+b \geq n \\ (B,b)=1}}^{n-1} \frac{n-B}{b}$$

Lemma 2. $\bar{\Psi}(m) = \frac{3m^2}{\pi^2} + O(m \log m)$

$$\frac{\varphi(1)}{1} + \frac{\varphi(2)}{2} + \dots + \frac{\varphi(m)}{m} = \frac{6m}{\pi^2} + O(\log^2 m)$$

$$\frac{\varphi(m+1)}{(m+1)^2} + \frac{\varphi(m+2)}{(m+2)^2} + \dots + \frac{\varphi(2m)}{(2m)^2} = \frac{6 \log 2}{\pi^2} + O\left(\frac{\log m}{m}\right).$$

Proof. The first is in Hardy and Wright². The third result is a special case of a result of D. H. Lehmer³. The second result can be derived from the first as follows:

$$\begin{aligned} & \frac{\varphi(1)}{1} + \frac{\varphi(2)}{2} + \dots + \frac{\varphi(m)}{m} \\ &= \frac{\bar{\Psi}(1)}{1} + \frac{\bar{\Psi}(2) - \bar{\Psi}(1)}{2} + \frac{\bar{\Psi}(3) - \bar{\Psi}(2)}{3} + \dots + \frac{\bar{\Psi}(m) - \bar{\Psi}(m-1)}{m} \\ &= \sum_{j=1}^{m-1} \frac{\bar{\Psi}(j)}{j(j+1)} + \frac{\bar{\Psi}(m)}{m} \\ &= \frac{3}{\pi^2} \left\{ \sum_{j=1}^{m-1} \frac{j^2}{j(j+1)} + \frac{m^2}{m} \right\} + O\left(\frac{m \log m}{m}\right) + \sum_{j=1}^{m-1} O\left(\frac{j \log j}{j(j+1)}\right) \\ &= \frac{3}{\pi^2} \left\{ 2m + O(\log m) \right\} + O(\log m) + O(\log^2 m) \\ &= \frac{6m}{\pi^2} + O(\log^2 m). \end{aligned}$$

Lemma 3. Let $\varphi(j, n)$ denote the number of integers $\leq j$ that are prime to n . Then

$$\varphi(j, n) = j \frac{\varphi(n)}{n} + O(\sqrt{n}).$$

Proof. Lehmer [3, p. 1188] proves that

$$|\varphi(j, n) - j \frac{\varphi(n)}{n}| < d(n)$$

where $d(n)$ is the number of divisors of n . To prove that $d(n) = O(\sqrt{n})$, say $d(n) < 4\sqrt{n}$ to be definite, we can use induction on the number of prime factors of n . Write

$n = p_1^{k_1} \cdots p_s^{k_s}$. If $s = 1$, we must prove that $(1+k_1)^2 < 16p_1^{k_1}$. It can be readily verified that

$$(*) \quad (1+k_1)^2 < 4p_1^{k_1}$$

for $p = 2$, and so a fortiori for $p > 2$. If $s = 2$, we must prove that

$$(1+k_1)^2(1+k_2)^2 < 16p_1^{k_1}p_2^{k_2}.$$

This is a consequence of (*). Next, if we assume the result for numbers with fewer than s prime factors, with $s \geq 3$, then we have

$$d(n) = (1+k_s)d\left(\frac{n}{p_s^{k_s}}\right) < (1+k_s) 4 \sqrt{\frac{n}{p_s^{k_s}}} < 4\sqrt{n}$$

provided that $(1+k_s)^2 < p_s^{k_s}$, and this is readily verified since $p_s \geq 5$.

Lemma 4. Let $S(j, n)$ denote the sum of the integers that are $\leq j$ and prime to n . Then

$$S(j, n) = \frac{j^2 \varphi(n)}{2n} + O(j\sqrt{n}).$$

Proof. We use induction on the number of prime factors of n . Let us write

$$n = p_1^{k_1} \cdots p_s^{k_s}, \quad n_1 = \frac{n}{p_s}, \quad j = qp_s + r, \quad 0 \leq r < p_s.$$

First, we prove the identity

$$(10) \quad S(j, n) = S(j, n_1) - p_s S(q, n_1).$$

This can be seen by observing that none of the numbers $p_s, 2p_s, \dots, qp_s$ are included in $S(j, n)$, but some of these are included in $S(j, n_1)$, namely, the integers in $1, 2, \dots, q$ that are prime to n_1 , each multiplied by p_s .

If $s = 1$, then (10) gives us

$$S(j, n) = S(j, 1) - p_s S(q, 1) = \frac{j(j+1)}{2} - p_1 \frac{q(q+1)}{2}.$$

Now, since

$$\frac{j^2 \varphi(n)}{2n} = \frac{1}{2} j^2 \left(1 - \frac{1}{p_1}\right),$$

we have

$$\begin{aligned} S(j, n) - \frac{j^2 \varphi(n)}{2n} &= \frac{1}{2} (j - p_1 q^2 - p_1 q + j^2/p_1) \\ &= rq + \frac{r}{2} + \frac{r}{2} \cdot \frac{r}{p_1} < pq + \frac{r}{2} + \frac{r}{2} = pq + r = j < j\sqrt{n}. \end{aligned}$$

Before turning to the rest of the induction argument, we handle the special case of integers n divisible by 2 and 3 but by no other primes. Thus $\frac{\varphi(n)}{n} = \frac{\varphi(6)}{6} = \frac{1}{3}$, and

$$S(j, n) = S(j, 6) = 1 + 5 + 7 + 11 + 13 + 17 + \dots$$

$$\begin{aligned} &= 6 + 18 + 30 + \dots + \left\{ 2\left[\frac{j}{6}\right] - 1 \right\} 6 + O(j) \\ &= 6 \left\{ 1 + 3 + 5 + \dots + \left(2\left[\frac{j}{6}\right] - 1\right) \right\} + O(j) \\ &= 6 \left[\frac{j}{6} \right]^2 + O(j) = \frac{j^2}{6} + O(j) = \frac{j^2 \varphi(n)}{2n} + O(j\sqrt{n}). \end{aligned}$$

We now use induction, assuming that the lemma holds for integers n with fewer than s prime factors; and we may presume that $p_s \geq 5$ because of the above special case. Applying (10) we have

$$\begin{aligned} S(j, n) &= S(j, n_1) - p_s S(q, n_1) \\ &= \frac{j^2 \varphi(n_1)}{2n_1} - \frac{p_s q^2 \varphi(n_1)}{2n_1} + O(j\sqrt{n_1}) + p_s O(q\sqrt{n_1}) \end{aligned}$$

But

$$\frac{j^2 \varphi(n)}{2n} = \frac{j^2 \varphi(n_1)}{2n_1} \left(1 - \frac{1}{p_s}\right) = \frac{j^2 \varphi(n_1)}{2n_1} - \frac{\varphi(n_1)}{2n_1} \left(q^2 p_s + 2qr + \frac{r^2}{p_s}\right)$$

Hence

$$S(j, n) - \frac{j^2 \varphi(n)}{2n} = \frac{\varphi(n_1)}{n_1} \left(qr + \frac{r^2}{2p_s}\right) + O(j\sqrt{n_1}) + O(p_s q \sqrt{n_1}).$$

$$(11) \quad T_1 = \sum_{\substack{b=1 \\ b=\left[\frac{n+1}{2}\right]}}^{n-2} \sum_{\substack{B=b+1 \\ (B,b)=1}}^{n-1} \frac{n-B}{b} .$$

In T_2 we have $2b < n$, $n \geq 2b-1$, so $B + b \geq n$ implies $B \geq n - b \geq b + 1$; therefore the sum ranges from $B = n-b$ to $B = n-1$ and we write

$$(12) \quad T_2 = \sum_{b=1}^{\left[\frac{n-1}{2}\right]} \sum_{\substack{B=n-b \\ (B,b)=1}}^{n-1} \frac{n-B}{b} .$$

We begin with (11). Replace B by $b+k$ in the inner sum; then

$$\sum_{\substack{B=b+1 \\ (B,b)=1}}^{n-1} \frac{n-B}{b} = \sum_{\substack{B=b+1 \\ (B,b)=1}}^n \frac{n-B}{b} = \sum_{k=1}^{n-b} \frac{n-b-k}{b} = \sum \frac{n-b}{b} - \sum \frac{k}{b}$$

$$= \frac{n-b}{b} \varphi(n-b, b) - \frac{1}{b} S(n-b, b)$$

$$= \frac{n-b}{b} \left\{ \frac{(n-b)\varphi(b)}{b} + O(\sqrt{b}) \right\} - \frac{1}{b} \left\{ \frac{(n-b)^2 \varphi(b)}{2b} + O[(n-b)\sqrt{b}] \right\} + O(1)$$

$$= \frac{(n-b)^2}{2b^2} \varphi(b) + O(\sqrt{n})$$

with the help of Lemmas 3 and 4.

Therefore, using Lemma 2, we find that

$$\begin{aligned} T_1 &= \sum_{\substack{b=1 \\ b=\left[\frac{n+1}{2}\right]}}^{n-2} \left\{ \frac{n^2}{2} \frac{\varphi(b)}{b^2} - n \frac{\varphi(b)}{b} + \frac{\varphi(b)}{2} + O(\sqrt{n}) \right\} \\ &= \frac{n^2}{2} \frac{6 \log 2}{\pi^2} - n \left(\frac{6n}{\pi^2} - \frac{3n}{\pi^2} \right) + \frac{1}{2} \left(\frac{3n^2}{\pi^2} - \frac{3n^2}{4\pi^2} \right) + O(n\sqrt{n}) \\ &= \frac{n^2}{\pi^2} \left(3 \log 2 - \frac{15}{8} \right) + O(n\sqrt{n}). \end{aligned}$$

To evaluate (12), we begin with the inner sum. Define r as the least positive residue of n modulo b , thus $r \equiv n \pmod{b}$, $1 \leq r \leq b$. Then we replace B by $n-r-i$ so that $(B, b) = 1$ becomes $(i, b) = 1$.

We have

$$\sum_{\substack{B=n-i \\ (B,b)=1}}^{n-1} \frac{n-B}{b} = \sum_{\substack{i=-r+1 \\ (i,b)=1}}^{b-r} \frac{r+i}{b} = \sum \frac{r}{b} + \sum \frac{i}{b}$$

(13)

$$= \frac{r \varphi(b)}{b} + \sum_{\substack{i=1 \\ (i,b)=1}}^{b-r} \frac{i}{b} + \sum_{\substack{i=-r+1 \\ (i,b)=1}}^{-1} \left(1 + \frac{i}{b}\right) - \varphi(r-1, b).$$

But, since

$$\begin{aligned} & \sum_{\substack{i=1 \\ (i,b)=1}}^{b-r} \frac{i}{b} + \sum_{\substack{i=-r+1 \\ (i,b)=1}}^{-1} \left(1 + \frac{i}{b}\right) = \sum_{\substack{i=1 \\ (i,b)=1}}^{b-r} \frac{i}{b} + \sum_{\substack{i=b-r+1 \\ (i,b)=1}}^{b-1} \frac{i}{b} \\ &= \sum_{\substack{i=1 \\ (i,b)=1}}^{b-1} \frac{i}{b} = \sum_{\substack{i=1 \\ (i,b)=1}}^{\left[\frac{b-1}{2}\right]} \frac{i}{b} + \sum_{\substack{i=1 \\ (i,b)=1}}^{\left[\frac{b-1}{2}\right]} \frac{b-1}{b} \\ &= \frac{\varphi(b)}{2}, \end{aligned}$$

we find that (13) becomes

$$\begin{aligned} & r \frac{\varphi(b)}{b} + \frac{\varphi(b)}{2} - \varphi(r, b) + O(1) \\ &= \frac{\varphi(b)}{2} + O(\sqrt{b}) \end{aligned}$$

by Lemma 3.

We use this result to evaluate the sum T_2 . We have

$$\begin{aligned} T_2 &= \sum_{b=1}^{\left[\frac{n-1}{2}\right]} \left\{ \frac{\varphi(b)}{2} + O(\sqrt{b}) \right\} \\ &= \frac{1}{2} \frac{3\left(\frac{n}{2}\right)^2}{\pi^2} + O(n\sqrt{n}) \text{ by Lemma 2} \\ &= \frac{3n^2}{8\pi^2} + O(n\sqrt{n}). \end{aligned}$$

Then

$$T_1 + T_2 = \frac{n^2}{\pi^2} \left\{ 3 \log 2 - \frac{3}{2} \right\} + O(n\sqrt{n}).$$

Thus the equation (9) becomes

$$\begin{aligned} \int_0^1 t(\alpha, \frac{1}{n}) d\alpha &= \frac{1}{n} \Psi(n-1) + \frac{2}{n} \frac{n^2}{\pi^2} \left\{ 3 \log 2 - \frac{3}{2} \right\} + O(\sqrt{n}) \\ &= \frac{1}{n} \left\{ \frac{3n^2}{\pi^2} + O(n \log n) \right\} + \frac{n}{\pi^2} \left\{ 6 \log 2 - 3 \right\} + O(\sqrt{n}) \\ &= \frac{6n \log 2}{\pi^2} + O(\sqrt{n}). \end{aligned}$$

Consequently, we have

$$\int_0^1 t(\alpha, \epsilon) d\alpha = \frac{6 \log 2}{\pi^2} \left(\frac{1}{\epsilon} \right) + O(\epsilon^{-1/2})$$

which proves (6).

To illustrate the higher dimensional cases, we take the two dimensional situation. The integer b is an admissible value for all pairs of real numbers lying in the region defined by the Cartesian product of any interval

$$I(0/b), I(1/b), \dots, I(b/b)$$

with any other. These intervals have length $2/n$ (disregarding what falls outside the unit interval of course) and the region defined by the Cartesian product has area $4/n^2$. Thus

$$\begin{aligned} \int_0^1 \int_0^1 t(a_1, a_2, \frac{1}{n}) da_1 da_2 &\geq 1 \frac{4}{n^2} + 2 \frac{4}{n^2} + \dots + \left[\frac{n^2}{4} \right] \frac{4}{n^2} \\ &\stackrel{\approx}{=} \frac{4}{n^2} \left\{ 1 + 2 + 3 + \dots + \left[\frac{n^2}{4} \right] \right\} \\ &\stackrel{\approx}{=} \frac{n^2}{8} \end{aligned}$$

because the region with actual (not admissible) value b is at most of area $\frac{4}{n^2}$. Thus the t -values larger than $\frac{n^2}{4}$ have been replaced by smaller values. In k dimensions, this argument proves the existence of a positive constant c , such that

$$\int_0^1 \int_0^1 \int_0^1 \cdots \int_0^1 t(a_1, \dots, a_k, \frac{1}{n}) da_1 \cdots da_k \geq c_1 n^k.$$

This integral is less than n^k , because by the pigeon hole procedure we can find a positive integer $m \leq n^k$ for any a_1, \dots, a_k such that ma_1, \dots, ma_k are all within $\frac{1}{n}$ of an integer.

Therefore, $t \leq n^k$ and then

$$\int_0^1 \int_0^1 \cdots \int_0^1 t \, da \cdots da_k \leq n^k.$$

This completes the proof of (5).

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